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STATE OF STRESS IN A PLANE CIRCULAR RING WEAKENED BY A SYSTEM OF CURVILINEAR CUTS*

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A plane problem of the theory of elasticity is studied for a circular ring weakened by a system of smooth, curvilinear cuts, with an arbitrary, self-equilibrating load acting at their edges. Integral representation of the complex stress potentials in terms of the displacement discontinuities at the cut contours in an infinite plane and the solution of the first fundamental problem for a solid circular ring are both used to reduce the problem in question to that of solving a system of singular equations. A limit equilibrium state of a ring containing an internal radial crack with a constant load acting at the crack edges or at the inner ring boundary is studied as an example.

The elastic equilibrium of a circular ring weakened by rectilinear cracks was studied for certain particular cases of crack distribution in /1–5/. The stresses in a ring with a curved crack situated on a concentric perimeter were determined in /6/.

1. Let the region S occupied by the ring be bounded by two concentric circles Γ_1 and Γ_2 of radii R_1 and R_2 $(R_1 < R_2)$ with the centers at the origin of the Cartesian xOy - coordinate system. The cuts are distributed along the contours L_n $(n = 1, 2, \ldots, N)$ referred to the local $x_nO_ny_n$ -coordinate systems the origins O_n of which are defined, in the fundamental coordinate system by the complex coordinates $z_n^\circ = x_n^\circ + iy_n^\circ$, with the O_nx_n axes forming the angles α_n with the Ox -axis (Fig.1).

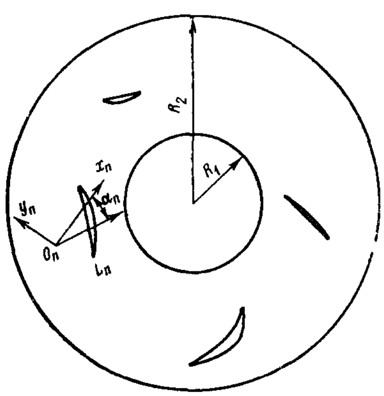

Fig.1

We shall assume that the contours Γ_1 and Γ_2 are stress-free and, that the self-equilibrated loads

$$N_n^\pm + iT_n^\pm = p_n(t_n), \tag{1.1}$$

$$z_n = x_n + iy_n,\ t_n = z_n \in L_n,\ n = 1, 2, \ldots, N$$

The complex potentials $\Phi_1(z)$ and $\Psi_1(z)$ which define the state of stress in an infinite plane caused by the displacement discontinuities $g_k(z_k)$ on the contours L_k $(k = 1, 2, \ldots, N)$, have the form /7/

$$\Phi_1(z) = \frac{1}{2\pi}\sum_{k=1}^{N} e^{i\alpha_k}\int_{L_k}\frac{g_k'(t)\,dt}{T_k - z} \tag{1.2}$$

$$T_k = te^{i\alpha_k} + z_k^\circ,\ z = x + iy$$

$$\Psi_1(z) = \frac{1}{2\pi}\sum_{k=1}^{N}\int_{L_k}\left[\frac{\overline{g_k'(t)\,dt}}{T_k - z}e^{-i\alpha_k} - \frac{T_k g_k'(t)\,dt}{(T_k - z)^2}e^{i\alpha_k}\right]$$

Let us assume that all N cuts in the infinite plane are situated within the region S. Using the formula /8/

$$\sigma_r - i\tau_{r\theta} = \Phi_1(z) + \overline{\Phi_1(z)} - e^{2i\theta}[z\Phi_1(z) + \Psi_1(z)]$$

and the functions (1.2), we obtain the combination of stresses $\sigma_r^j - i\tau_{r\theta}^j$ generated in the infinite plate at the circles Γ_1 $(j = 1)$ and Γ_2 $(j = 2)$ by the discontinuities $g_k(z_k)$ $(k = 1, 2, \ldots, N)$ of the displacements

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$$\sigma_r{}^j - i\tau_{r\theta}^j = \frac{1}{2\pi}\sum_{k=1}^{N}\int_{L_k}[F_{1k}^j(t,\sigma)g_k'(t)\,dt + F_{2k}^j(t,\sigma)\overline{g_k'(t)}\,\overline{dt}] \tag{1.3}$$

$$F_{1k}^j(t,\sigma) = e^{i\alpha_k}\left[\frac{1}{T_k - R_j\sigma} - \frac{\sigma^2(R_j\bar\sigma - \bar T_k)}{(T_k - R_j\sigma)^2}\right]$$

$$F_{2k}^j(t,\sigma) = e^{-i\alpha_k}\left[\frac{1}{\bar T_k - R_j\bar\sigma} - \frac{\sigma^2}{T_k - R_j\sigma}\right], \quad \sigma = e^{i\theta}, \quad j = 1, 2$$

Having solved the auxilliary problem for a circular ring for the case when the loads are equal in magnitude but opposite in direction with respect to the stresses (1.3) act at its boundary, we determine the complex potentials /8/

$$\Phi_2(z) = \sum_{s=-\infty}^{\infty} a_s z^s, \quad \Psi_2(z) = \sum_{s=-\infty}^{\infty} b_s z^s$$

The coefficients a_s and b_s can be written in the form

$$a_s = \frac{1}{2\pi d^s}\sum_{k=1}^{N}\int_{L_k}\{a_{sk}^1(t)g_k'(t)\,dt + a_{sk}^2(t)\overline{g_k'(t)}\,\overline{dt}\}$$

$$b_s = \frac{1}{2\pi d^s}\sum_{k=1}^{N}\int_{L_k}\{b_{sk}^1(t)g_k'(t)\,dt + b_{sk}^2(t)\overline{g_k'(t)}\,\overline{dt}\},$$

$$s = 0, \pm 1, \pm 2, \ldots$$

$$a_{0k}^j(t) = \frac{1}{2(1-m^2)}\{B_{0kj}^2(t) - m^2 B_{0kj}^1(t)\}$$

$$a_{1k}^j(t) = \frac{1}{d\varepsilon_2(1-m^4)}\{\overline{B_{1k\nu}^2(t)} - m^3\overline{B_{1kj}^1(t)}\}, \quad a_{-1k}^j(t) = 0$$

$$a_{sk}^j(t) = d^{-s}\{\beta_{1s}B_{skj}^1(t) + \beta_{2s}B_{skj}^2(t) + \beta_{3s}\overline{B_{-sk\nu}^1(t)} + \beta_{4s}\overline{B_{-sk\nu}^2(t)}\};$$

$$s = \pm 2, \pm 3, \ldots$$

$$b_{sk}^j(t) = \varepsilon_1^{-2(s+1)}a_{-(s+2)k}^\nu(t) - (1+s)\varepsilon_1^2 a_{(s+2)k}^j(t) - \varepsilon_1^{-s}B_{(s+2)k\nu}^1(t)$$

$$s = 0, \pm 1, \pm 2, \ldots, \quad j, \nu = 1, 2, \quad j \neq \nu$$

$$B_{sk\nu}^j(t) = -\frac{1}{2\pi}\int_0^{2\pi}F_{\nu k}^j(t,\sigma)e^{-is\theta}\,d\theta; \quad s = 0, \pm 1, \pm 2, \ldots,$$

$$j, \nu = 1, 2$$

$$\beta_s{}^1 = -m^{2-s}\beta_s{}^2, \quad \beta_s{}^2 = \frac{(1+s)(1-m^2)}{\beta_0}, \quad \beta_s{}^3 = m^{2+s}\beta_s{}^4$$

$$\beta_s{}^4 = -\frac{1-m^{-2s+2}}{\beta_0}, \quad \beta_0 = [(1-s^2)(1-m^2)^2 -$$

$$(1-m^{2s+2})(1-m^{-2s-2})]\,\varepsilon_2{}^s$$

$$\varepsilon_1 = \frac{R_1}{d}, \quad \varepsilon_2 = \frac{R_2}{d}, \quad m = \frac{R_1}{R_2}, \quad d = R_2 - R_1$$

The complex potentials $\Phi(z)$ and $\Psi(z)$ describing the state of stress in a circular ring with a free boundary and the discontinuity displacements $g_k(z_k)$ specified at the contours L_k, can be written in the form

$$\Phi(z) = \Phi_1(z) + \Phi_2(z), \; \Psi(z) = \Psi_1(z) + \Psi_2(z)$$

Using the above relations to fulfil the boundary conditions (1.1), we obtain a system of N singular integral equations for the functions $g_k'(z_k)$ $(k = 1, 2, \ldots, N)$

$$\sum_{k=1}^{N}\int_{L_k}[K_{nk}(t,t')g_k'(t)\,dt + M_{nk}(t,t')\overline{g_k'(t)}\,\overline{dt}] = \pi p_n(t_n') \tag{1.4}$$

$$t_n' \in L_n, \; n = 1, 2, \ldots, N$$

$$K_{nk}(t,t')=\frac{e^{i\alpha_k}}{2}\left[\frac{1}{T_k-T_n'}+\frac{e^{-2i\alpha_n}}{\overline{T}_k-\overline{T}_n'}\frac{\overline{dt_n'}}{dt_n'}\right]+\frac{1}{2}\sum_{s=-\infty}^{\infty}\left[a_{sk}^1(t)\,T_n'^s+\overline{a_{sk}^2(t)}\,\overline{T}_n'^s+e^{-2i\alpha_n}\frac{\overline{dt_n'}}{dt_n'}\{s\overline{a_{sk}^2(t)}\,T_n'\overline{T}_n'^{s-1}+\overline{b_{sk}^2(t)}\,\overline{T}_n'^s\}\right]$$

$$M_{nk}(t,t')=\frac{e^{-i\alpha_k}}{2}\left[\frac{1}{\overline{T}_k-\overline{T}_n'}-\frac{(T_k-T_n')\,e^{-2i\alpha_n}}{(\overline{T}_k-\overline{T}_n')^2}\frac{\overline{dt_n'}}{dt_n'}\right]+$$

$$\frac{1}{2}\sum_{s=-\infty}^{\infty}\left[a_{sk}^1(t)\,\overline{T_n'^s}+a_{sk}^2(t)\,T_n'^s+e^{-2i\alpha_n}\frac{\overline{dt_n'}}{dt_n'}\{sa_{sk}^1(t)\,T_n'\overline{T}_n'^{s-1}+\overline{b}_{sk}^1(t)\,\overline{T}_n'^s\},\quad T_n'=t'e^{i\alpha_n}+z_n^0.\right.$$

In the case of internal cracks L_n, the solution of the system (1.4) must satisfy the conditions

$$\int_{L_n} g_n'(t)\,dt=0,\quad n=1,2,\dots,N$$

which ensure the singlevaluedness of the displacements on traversing the contours L_n.

Thus the problem of determining the stress-strain state of a circular ring weakened by a system of curvilinear cracks is reduced to that of solving singular integral equations.

2. As an example we shall consider a limiting case of the state of equilibrium of a circular ring weakened by a rectilinear crack of length $2l$, placed along the Ox-axis. We shall assume that the center of the crack is situated at the distance h from the inner boundary of the ring. In this case we must put, in the fomulas (1.4),

$$n=k=1,\quad \alpha_1=0,\quad \frac{\overline{dt_1}}{dt_1}=1,\quad z_1^0=R_1+h$$

and solve one singular integral equation in order to determine the function $g_1'(x_1)$. We obtain

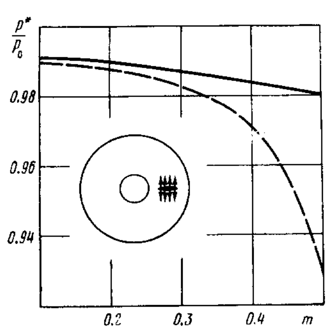

Fig.2

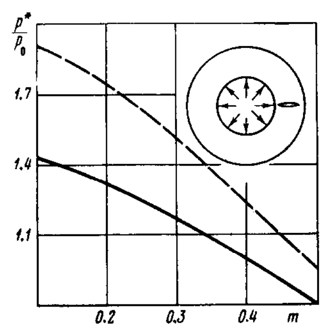

Fig.3

a numerical solution of the integral equation using the method of mechanical quadratures and the Gauss—Chebyshev quadrature formulas /9/. We find the stress intenity factors as the crack tips using the known function $g_1(t_1)$

$$k_1^{\pm}-ik_2^{\pm}=\lim_{t_1\to l^{\pm}}\sqrt{2|t_1-l^{\pm}|}\,g_1'(t_1)$$

where the lower signs refer to the beginning of the crack $t_1=l^-$ and the upper signs to the end of the crack $t_1=l^+$.

Figure 2 depicts the dependence of the magnitude of the critical load p^* applied to the crack edges, divided by p_0 (p_0 is the value of the critical load acting at the crack in an infinite plane /10/) on the quantity m, for $\lambda=0.6, \varepsilon_1=3$. The solid line refers to the left end of the crack, and the broken line to the right end. Analysis of the graphs indicates that for the given position of the crack the quantity p^* shows little change for the left end of the crack when the parameter m varies. For the right end of the crack p^* decreases with increasing m when $m>0.4$. When $m<0.2$, the value of p^* for a circular ring is practically equal to that for an infinite plane weakened by a circular hole and a crack /9/.

When normal pressure acts on the inner boundary of the ring, then the function $p(t_1)$ equal in magnitude but opposite in sign to the normal stresses appearing along the crack in a solid ring, has the form /8/

$$p(t_1) = -p\left[\frac{m^2}{1-m^2} - \frac{R_1^2}{1-m^2}\frac{1}{t_1^2}\right]$$

Figure 3 depicts, for the last case of the loading, the dependence of p^*/p_0 on the parameter m, for $\varepsilon = 3, \lambda = 0.6$. The graphs show that p^* decreases with increasing m for both the right and the left end of the crack. The crack propagates from the left tip over the whole interval of variation in m.

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